Math 250 3.3 Basic Differentiation Rules and Derivatives of $\sin x$ **and** $\cos x$ (from 3.5)

Objectives

1) Find derivatives using (shortcuts! ☺) rather than the definition of the derivative (limits! ☹)
   a. Constant Rule $\frac{d}{dx}[c]=0$
   b. Power Rule $\frac{d}{dx}[x^n]=nx^{n-1}$
   c. Constant Multiple Rule $\frac{d}{dx}[c\cdot f(x)]=c\cdot f'(x)$
   d. Sum and Difference Rules $\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$
   e. **CAUTION**: There is NOT a parallel rule for the derivatives of products or quotients!
2) Find higher-order derivatives by repeated use of differentiation rules.
   a. The second derivative f'', is the derivative of the first derivative: $f''(x)=\frac{d}{dx}[f'(x)]$
   b. The third derivative f''', is the derivative of the second derivative: $f'''(x)=\frac{d}{dx}[f''(x)]$, etc.
   c. Higher-order derivatives can also use the $\frac{d}{dx}$ notation, but notice that the operator $\frac{d}{dx}$ is applied each time the derivative is taken, so exponents in the numerator are different from exponents in the denominator. $f''(x)=\frac{d^2y}{dx^2}$, $f'''(x)=\frac{d^3y}{dx^3}$
3) Find derivatives of sin(x), cos(x)
   a. Two special limits: memorize!
      i. $\lim\limits_{x\to 0}\frac{\sin x}{x}$
      ii. $\lim\limits_{x\to 0}\frac{1-\cos x}{x}$
   b. $\frac{d}{dx}\sin x$
   c. $\frac{d}{dx}\cos x$

CAUTION: Finding the derivative using the definition requires algebra and limits. We want shortcuts to avoid using this process!
If the instructions say to "find derivative using the limit process" or "differentiate using the definition of derivative", we MUST use limits.
If the instructions say "find derivatives" or "differentiate", we can and should use the rules and shortcuts.

MUST MEMORIZE ALL THE DERIVATIVE RULES!

Recall: Definition of Derivative

$$m_{TAN}(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Examples and Practice:

1) Graph $f(x) = x$ and use the graph find the slope of the tangent line at $(2, f(2))$.

2) Graph $f(x) = 3$ and use the graph find the slope of the tangent line at $(2, f(2))$. What general statement can we make about $f(x) = c$ for any constant c?

3) Use the definition of limit and the Binomial Theorem to find the derivative of $f(x) = x^4$. What general statement can we make about $f(x) = x^n$?

We have no proof yet, but the Power Rule shown for positive integers also works for any real number except 0. $f(x) = x^0 = 1$ uses the Constant Rule.

4) Use the Power rule to find the following derivatives
   a. $f(x) = x^{23}$
   b. $f(x) = x^{-3}$
   c. $f(t) = t^{\pi}$
   d. $s(t) = t^{1/3}$
   e. $f(x) = \sqrt[4]{x}$

5) Use the definition to find the derivative of $g(x) = c \cdot f(x)$. What is the difference between this rule and the rule for $f(x) = c$?

6) Use the definition to differentiate $h(x) = f(x) + g(x)$, and draw a conclusion about the derivatives of $h(x) = f(x) + g(x)$ and $h(x) = f(x) - g(x)$.

7) Find first and second derivatives derivatives, simplifying first if necessary.

   a. $y = 5x^3 - \frac{7}{8}x^2 + 3x$

   b. $h(x) = \dfrac{x^3 + 4x^2 - 6}{x^2}$

   c. $s(t) = 3t(6t - 5t^2)$

8) Find the equation of the tangent line to $g(x) = 3(5 - x)^2$ at $(4,3)$

9) Determine the points, if any, at which the graph of $f(x) = x^3 - x$ has a horizontal tangent line.

10) Using $f(x) = \frac{5}{(2x)^3}$, find $f'(-4)$ and sketch a graph demonstrating what this means.

11) Use numerical evidence to find the limit

a. $\lim_{x \to 0} \frac{\sin x}{x}$

b. See notes online for analytic $\lim_{x \to 0} \frac{\sin x}{x}$ by the Squeeze theorem

c. $\lim_{x \to 0} \frac{1 - \cos x}{x}$

d. See notes online for analytic $\lim_{x \to 0} \frac{1 - \cos x}{x}$ using $\lim_{x \to 0} \frac{\sin x}{x}$

12) Recall the identities for

a. $\sin(A + B)$

b. $\cos(A + B)$

13) Use the definition to find the derivative when $f(x) = \sin x$

14) Use the definition to derive the derivative of $f(x) = \cos x$

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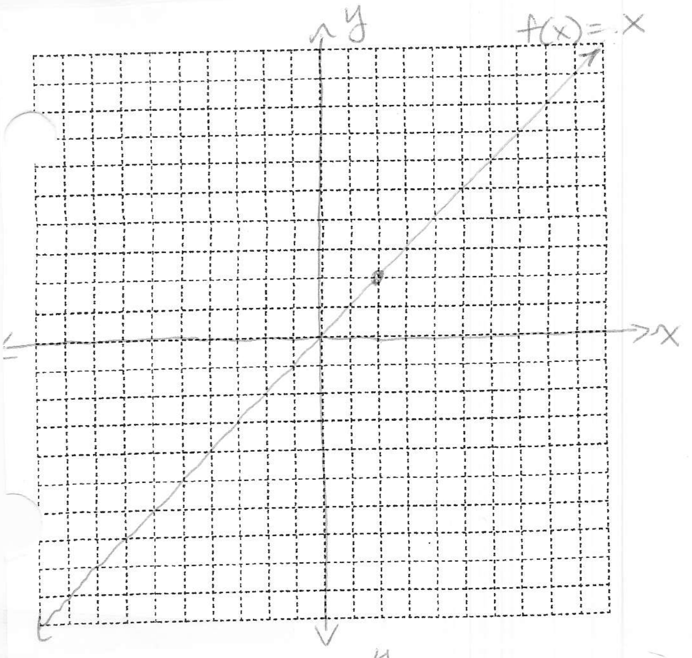


① Graph $f(x)=x$, find slope (and equation) of tangent at $(2, f(2))$ using graph.

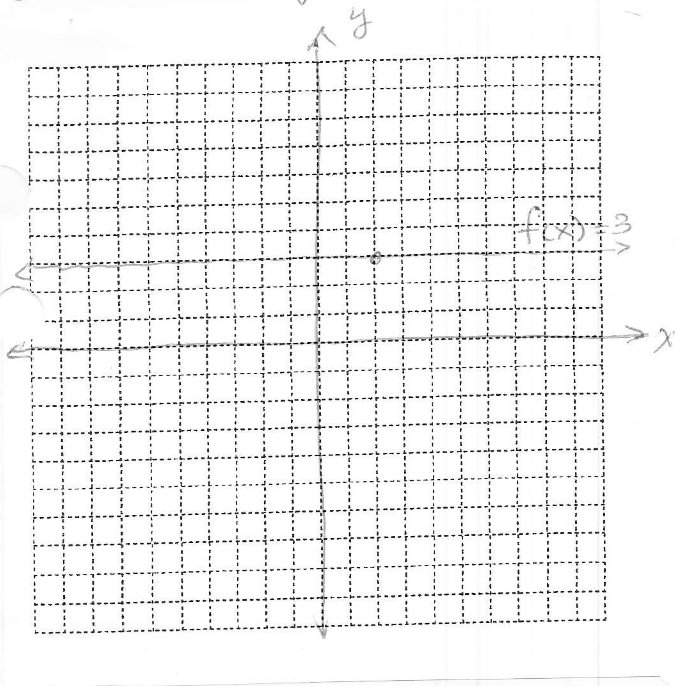
$f(2)=2$ so $(2, f(2)) = (2,2)$

tangent line is <u>same</u> as $f(x)$.
tangent line has $\boxed{\text{slope } 1}$

$y = mx + b \iff y = 1x + 0$

$y - 2 = 1(x-2)$

$\boxed{y = x}$



② Graph $f(x)=3$ Find slope and Find equation of tangent at $(2, f(2))$ Using graph

$f(2)=3$ so $(2, f(2)) = (2,3)$

tangent line is <u>same</u> as $f(x)$.
tangent line is <u>horizontal</u> $\Rightarrow$ $\boxed{\text{slope is } 0}$

$\boxed{y=3}$

eneral form.

Any function $f(x)=c$ is a horizontal line, with all of its tangents being horizontal and having slope 0.

Find $f'(x)$ if $f(x)=c$ (a constant).

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$
$$= \lim_{\Delta x \to 0} 0$$
$$= 0. \qquad \boxed{f'(x) = 0}$$

Constant Rule $\frac{d}{dx}[c] = 0$

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$$f(x) = x^4.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3)}{\cancel{h}}$$

$$= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3)$$

$$= 4x^3 + 0 + 0 + 0$$

$$= \boxed{4x^3}$$

Recall Binomial Theorem?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

2nd term of Binomial Expansion survives limit process

$$\boxed{\frac{d}{dx}[x^n] = n \cdot x^{n-1} \quad \text{Power Rule}}$$

Example 1 using power rule:

$$f(x) = x = x^1$$

$$f'(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

$$f'(x) = 1$$

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Does the power rule work if the exponent is not a positive integer (exponent from the Binomial Theorem)?

Yes, but we won't prove it until later.

Negative Integer Example

$f(x) = \frac{1}{x}$ Find $f'(x)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ defn of derivative

$$= \lim_{\Delta x \to 0} \left(\frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \right) \frac{\cdot x(x+\Delta x)}{\cdot x(x+\Delta x)}$$ subst $f(x+\Delta x)$ and $f(x)$; clear complex frac.

$$= \lim_{\Delta x \to 0} \frac{x - (x+\Delta x)}{x(\Delta x)(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x} - \cancel{x} - \cancel{\Delta x}}{\cancel{x(\Delta x)}(x+\Delta x)}$$ dist neg; combine like terms; cancel $\frac{\Delta x}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)}$$ take limit by subst $\Delta x = 0$.

$$\boxed{f'(x) = \frac{-1}{x^2}} = -x^{-2}$$ simplify

By power rule:

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}\left[x^{-1}\right] = -1\, x^{-1-1} = -1x^{-2} = \frac{-1}{x^2} \checkmark$$

Power Rule also works when
- exponent is a fraction (rational number)
- exponent is an irrational number.

but we don't prove this until later.

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(4) Find derivatives

a) $f(x) = x^{23}$

$f'(x) = 23\,x^{23-1}$ Power Rule

$\boxed{f'(x) = 23x^{22}}$

b) $f(x) = x^{-3}$

$f'(x) = -3x^{-3-1}$ Power rule exponent is always subtract 1.

$\boxed{f'(x) = -3x^{-4}}$

c) $f(t) = t^{\pi}$

$\boxed{f'(t) = \pi \cdot t^{\pi-1}}$ $\pi - 1$ cannot be simplified — do not approximate by rounding!

d) $s(t) = t^{1/3}$

$s'(t) = \frac{1}{3}t^{1/3-1}$ Power Rule

$\boxed{s'(t) = \frac{1}{3}t^{-2/3}}$ OR $\boxed{s'(t) = \frac{1}{3t^{2/3}}}$

e) $f(x) = \sqrt[4]{x}$

$f(x) = x^{1/4}$ rewrite using exponent

$f'(x) = 1/4\,x^{1/4-1}$ power rule

$\boxed{f'(x) = \frac{1}{4}x^{-3/4}}$

or $\boxed{f'(x) = \frac{1}{4x^{3/4}}}$

or $\boxed{f'(x) = \frac{1}{4\sqrt[4]{x^3}}}$

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(5)

$g'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$	definition of derivative
$= \lim_{h\to 0} \dfrac{c\cdot f(x+h) - c\cdot f(x)}{h}$	substitute $f(x+h)$ and $f(x)$
$= \lim_{h\to 0} \dfrac{c\,[f(x+h)-f(x)]}{h}$	factor out c.
$= c\cdot\left[\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}\right]$	property of limits $\lim_{x\to a} c\cdot f(x) = c\cdot \lim_{x\to a} f(x)$
$= c\cdot f'(x).$	definition of derivative $f'(x)$

$$\boxed{\text{Constant Multiple Rule} \quad \frac{d}{dx}[c\cdot f(x)] = c\cdot f'(x)}$$

What is the difference between taking the derivative of $h(x) = c\cdot f(x)$ and the derivative of $h(x) = c$?

This is a constant multiple. It remains in the derivative.

This is a constant. Its derivative is 0.

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6 $\frac{d}{dx}[f(x)+g(x)]$

$= \lim_{h\to 0} \frac{f(x+h)+g(x+h) - [f(x)+g(x)]}{h}$ definition of derivative

$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$ dist neg

$= \lim_{h\to 0} \left\{ \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right\}$ algebra — collect f together collect g together

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ property of limits

$\lim_{x\to c} (h(x)+k(x))$

$= \lim_{x\to c} h(x) + \lim_{x\to c} k(x)$

$= f'(x) + g'(x)$ Substitute definition of derivative $f'(x)$ and $g'(x)$.

$\boxed{\text{Sum Rule } \frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)}$

$\frac{d}{dx}[f(x)-g(x)]$ would be the same except subtracted.

Difference Rule $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

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(7) Find first and second derivatives.

a) $y = 5x^3 - \frac{7}{8}x^2 + 3x$

$y'(x) = 5 \cdot 3x^2 - \frac{7}{8} \cdot 2x^1 + 3 \cdot 1x^0$

constant multiple | power rule | constant multiple | power rule | constant multiple | power rule

$\boxed{y'(x) = 15x^2 - \frac{7}{4}x + 3}$ alternate notation $\boxed{\frac{dy}{dx} = 15x^2 - \frac{7}{4}x + 3}$

$y''(x) =$ differentiate $y'(x)$

$y''(x) = 15 \cdot 2x^1 - \frac{7}{4} \cdot 1x^0 + 0$

constant multiple | power rule | constant multiple | power rule | constant

$\boxed{y''(x) = 30x - \frac{7}{4}}$ alternate notation $\boxed{\frac{d^2y}{dx^2} = 30x - \frac{7}{4}}$

b) $h(x) = \frac{x^3 + 4x^2 - 6}{x^2}$ variables in both numerator and denom ☹ CANNOT take derivatives of numerator and denominator separately. (see 3.4)

Simplify

$h(x) = \frac{x^3}{x^2} + \frac{4x^2}{x^2} - \frac{6}{x^2}$ or $x^{-2}(x^3 + 4x^2 - 6)$

$h(x) = x + 4 - 6x^{-2}$

$h'(x) = 1x^0 + 0 - 6 \cdot (-2)x^{-3}$

power rule | constant | constant multiple | power rule

$\boxed{h'(x) = 1 + 12x^{-3} = 1 + \frac{12}{x^3} = \frac{x^3 + 12}{x^3}}$

Though all 3 forms are valid, the last will be most useful in later

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(7) cont, b) cont

$h''(x) =$ derivative of $h'(x)$

$h''(x) = 0 + 12 \cdot (-3)x^{-4}$

constant　constant multiple　power rule

$$\boxed{h''(x) = -36x^{-4} = \frac{-36}{x^4}}$$

c) $s(t) = 3t(6t - 5t^2)$

variable · variable

CANNOT take derivatives of each factor separately (see 3.4)

simplify by distributing

$s(t) = 18t^2 - 15t^3$

$s'(t) = 18 \cdot 2t^1 - 15 \cdot 3t^2$

constant multiples　power rules

$$\boxed{s'(t) = 36t - 45t^2}$$

$s''(t) =$ derivative of $s'(t)$

$s''(t) = 36 \cdot 1t^0 - 45 \cdot 2t^1$

constant multiples　power rules

$$\boxed{s''(t) = 36 - 90t}$$

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(8) Write the equation of the line tangent to $f(x) = 3(5-x)^2$ at $(4,3)$.

Step 1: Find $f'(x)$.

$f(x) = 3(5-x)^2$ — Simplify f(x) first

$= 3(25-10x+x^2)$

$= 75-30x+3x^2$

$f'(x) = -30+6x$ — Power and constant multiple rules

Step 2: Evaluate $f'(4)$ to find the slope of the tangent line at $(4,3)$.

$f'(4) = -30+6(4) = -6$

$m = -6$

Step 3: Use point-slope formula for equation of a line.

$y-y_1 = m(x-x_1)$

$y-3 = -6(x-4)$

Step 4: Write equation in general form.

$y-3 = -6x+24$

$\boxed{6x+y-27=0}$

(9) Determine the points (if any) at which the graph has a horizontal tangent line. $y = x^3 - x$

Step 1: tangent line ⇒ derivative

$y'(x) = 3x^2-1$

Step 2: horizontal ⇒ slope = 0

$y'(x) = 0$

$3x^2-1=0$

$3x^2=1$

$x^2=\frac{1}{3}$

$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

when $x = \frac{\sqrt{3}}{3}$ $\quad y = \left(\frac{\sqrt{3}}{3}\right)^3 - \frac{\sqrt{3}}{3} = \frac{3\sqrt{3}}{27} - \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{9} - \frac{3\sqrt{3}}{9} = -\frac{2\sqrt{3}}{9}$

$\boxed{\text{point } \left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{9}\right)}$

when $x = -\frac{\sqrt{3}}{3}$ $\quad y = \left(-\frac{\sqrt{3}}{3}\right)^3 - \left(-\frac{\sqrt{3}}{3}\right) = -\frac{3\sqrt{3}}{27} + \frac{\sqrt{3}}{3} = -\frac{\sqrt{3}}{9} + \frac{3\sqrt{3}}{9} = \frac{2\sqrt{3}}{9}$

$\boxed{\text{point } \left(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)}$

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10 $f(x) = \dfrac{5}{(2x)^3}$, find $f'(-4)$, sketch graph of meaning.

Simplify $f(x) = \dfrac{5}{8x^3} = \dfrac{5}{8}x^{-3}$

differentiate $f'(x) = \underset{\substack{\uparrow \\ \text{constant multiple}}}{\dfrac{5}{8}} \cdot (-3) \cdot \underbrace{x^{-3-1}}_{\text{power rule}}$

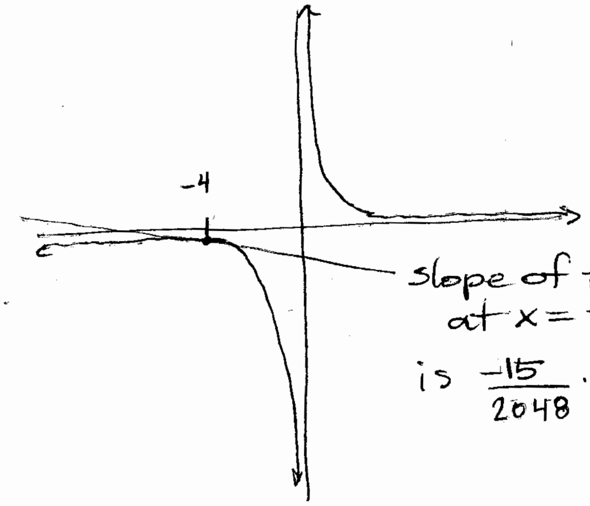
$f'(x) = \dfrac{-15}{8}x^{-4} = \dfrac{-15}{8x^4}$

Evaluate when $x = -4$

$f'(-4) = \dfrac{-15}{8(-4)^4} = \dfrac{-5}{2048} \approx -.0071976967$

negative, but very close to 0.

Graph of f(x)

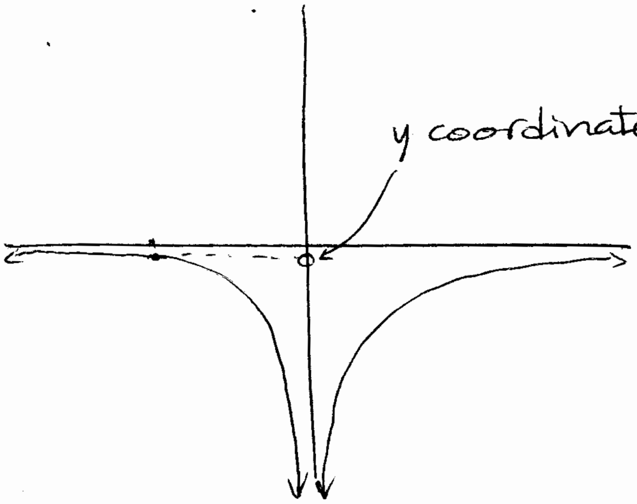


slope of tangent to $f(x)$ at $x = -4$ is $\dfrac{-15}{2048} = f'(-4)$

y coord of $f(x)$ at $x=-4$

$f(-4) = \dfrac{5}{(-8)^3} = \dfrac{-5}{512}$

Graph of $f'(x)$



y coordinate of f' at $x = -4$ is $\dfrac{-15}{2048}$.

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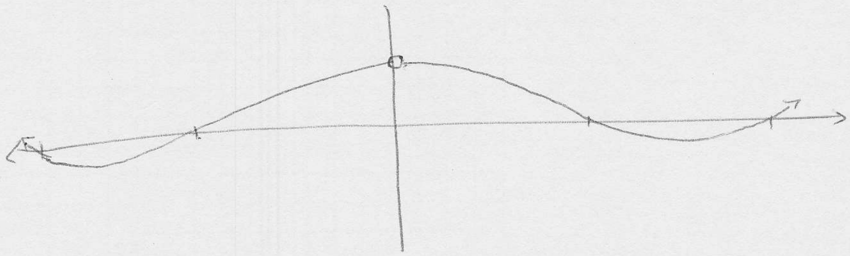
(11) a) $\lim_{x \to 0} \frac{\sin x}{x}$



$f(0) = \frac{\sin(0)}{0} = \frac{0}{0}$ indeterminate

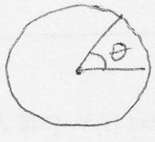
x	$-.1$	$-.01$	$-.001$	0	$.001$	$.01$	$.1$
$f(x)$	$.99833$	$.99998$	1.00000		1.00000	$.99998$	$.99833$

$\lim_{x \to 0} f(x) \approx \boxed{0}$

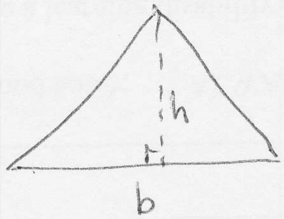
(11) b)

Proof of $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$ ← Memorize this limit!

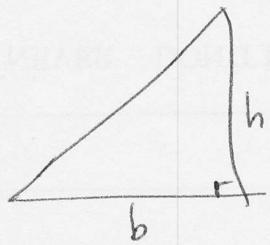
Easier to see as $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$



sector of a ⊙ has area $\frac{\theta}{2}$ (244/104)

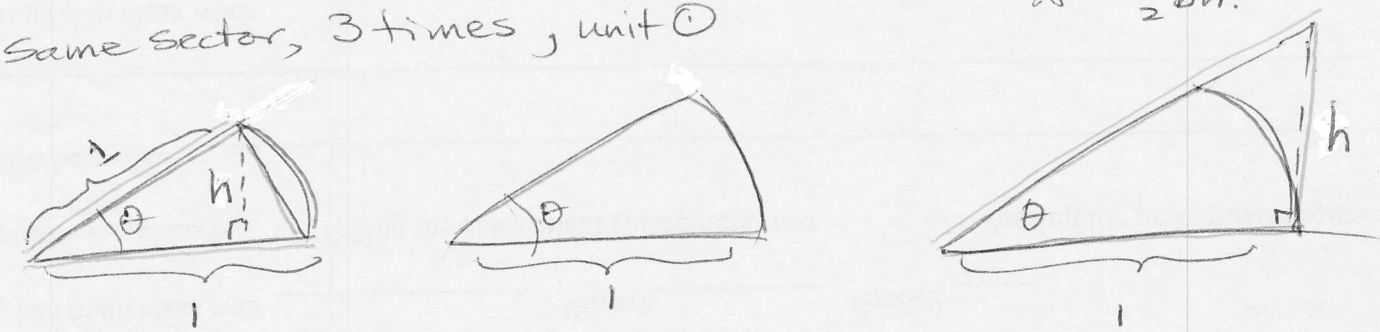


Area of Δ is
$A = \frac{1}{2}bh$



Area of Δ is
$A = \frac{1}{2}bh.$

Same sector, 3 times, unit ⊙



$\frac{y}{r} = \frac{h}{1} = \sin\theta$

$h = \sin\theta$

$\frac{y}{x} = \frac{h}{1} = \tan\theta$

$$\text{Area } \Delta \le \text{Area sector} \le \text{Area } \Delta$$

$$\frac{1}{2}\cdot 1 \cdot \sin\theta \le \frac{\theta}{2} \le \frac{1}{2}\cdot 1 \cdot \tan\theta$$

$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2}$$

Mult by $\frac{2}{\sin\theta}$: $\frac{\sin\theta}{2}\cdot\frac{2}{\sin\theta} \le \frac{\theta}{2}\cdot\frac{2}{\sin\theta} \le \frac{\tan\theta}{2}\cdot\frac{2}{\sin\theta}$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

Take reciprocals + reverse $\le$: $\cos\theta \le \frac{\sin\theta}{\theta} \le 1$

Take limits $\lim_{\theta\to 0}\cos\theta \le \lim_{\theta \to 0}\frac{\sin\theta}{\theta} \le \lim_{\theta\to 0} 1$

$1 \le \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \le 1$ Squeeze.

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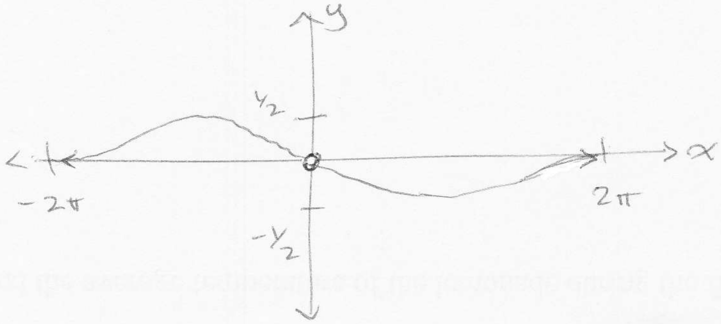
(1) a) $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

$f(x) = \dfrac{\cos x - 1}{x} = \dfrac{\cos(x) - 1}{x}$



$f(0) = \dfrac{\cos(0) - 1}{0} = \dfrac{1-1}{0} = \dfrac{0}{0}$ indeterminate

x	$-.1$	$-.01$	$-.001$	0	$.001$	$.01$	$.1$
$f(x)$	$-.05$	$-.005$	$-.00049$		$-.00049$	$-.005$	$-.05$

as $x \to 0$ from left $\rightarrow$, $y \to 0$ $\leftarrow$ as $x \to 0$ from right, $y \to 0$

$\lim\limits_{x \to 0} f(x) \approx \boxed{0}$

11 d) Proof of $\boxed{\lim_{x \to 0} \frac{1-\cos x}{x} = 0}$ ← Memorize this limit!

$$\lim_{x \to 0} \frac{1-\cos x}{x}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$ Mult by 1

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$ Foil combine

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$ Pythag identity (104/244)

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$ Separate

$$= \underbrace{\lim_{x \to 0} \frac{\sin x}{x}}_{\text{special limit}} \cdot \underbrace{\lim_{x \to 0} \frac{\sin x}{1+\cos x}}_{\text{evaluate}}$$

$$= 0 \cdot \frac{\sin(0)}{1+\cos(0)}$$

$$= 0 \cdot \frac{0}{1}$$

$$= 0 \cdot 0$$

$$= 0.$$

12 a) $\sin(A+B) = \sin A \cos B + \sin B \cos A$

b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

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13

Differentiate $f(x) = \sin x$ using definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

defn of $f'(x)$.

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

replace $f(x+h)$ by $\sin(x+h)$ and $f(x)$ by $\sin(x)$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin x}{h}$$

$\sin(A+B)$ identity with $A=x$ and $B=h$.

$$= \lim_{h \to 0} \frac{\sin(h)\cos(x) + \sin(x)\cos(h) - \sin x}{h}$$

collect terms containing $\sin x$.

Focus on grouping terms to create known limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

only we'll need slightly different variables:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0$$

$$= \lim_{h \to 0} \frac{\sin(h)}{h} \cdot \cos(x) + \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h}$$

properties of limits

$$= \left[\lim_{h \to 0} \frac{\sin(h)}{h}\right] \cdot \left[\lim_{h \to 0} \cos(x)\right] + \left[\lim_{h \to 0} \sin(x)\right] \cdot \left[\lim_{h \to 0} \frac{\cos(h) - 1}{h}\right]$$

factor out $\sin(x)$, prop. of limits

$$= \quad " \qquad " \quad - \quad " \quad \left[\lim_{h \to 0} \frac{1 - \cos(h)}{h}\right]$$

factor out -1

$$= \left[\lim_{h \to 0} \frac{\sin(h)}{h}\right] \cdot \left[\lim_{h \to 0} \cos(x)\right] - \left[\lim_{h \to 0} \sin(x)\right] \cdot \left[\lim_{h \to 0} \frac{1 - \cos(h)}{h}\right]$$

known limits,

$$= 1 \cdot \cos(x) - \sin(x) \cdot 0 = \boxed{\cos x}$$

arith.

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(14) Find derivative of $f(x) = \cos(x)$ using definition.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

subst $f(x) = \cos(x)$
& $f(x+h) = \cos(x+h)$

$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$\cos(A+B)$ identity
$A = x$ & $B = h$.

$$= \lim_{h\to 0} \left[-\frac{\sin(x)\sin(h)}{h} + \frac{\cos(x)\cos(h) - \cos(x)}{h} \right]$$

collect $\cos x$

$$= \lim_{h\to 0} \left[-\sin(x) \cdot \frac{\sin(h)}{h} + \cos(x)\left(\frac{\cos(h) - 1}{h}\right) \right]$$

collect known limits

$$= \left[\lim_{h\to 0} -\sin(x)\right] \cdot \left[\lim_{h\to 0} \frac{\sin(h)}{h}\right] + \left[\lim_{h\to 0} \cos(x)\right] \cdot \left[\lim_{h\to 0} \frac{\cos(h)-1}{h}\right]$$

properties of limits

$$= -\sin(x) \cdot 1 + \cos(x) \cdot 0$$

known limits

$$= \boxed{-\sin(x)}$$

arithmetic